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On a Local Energy Decay of Solutions of a Dissipative Wave Equation

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§1. Introduction.

This study is concerned with a local energy decay property of solutions to the following initial boundary value problem of the dissipative wave equation :

$$(D)\begin{cases} u_{tt}+u_t-\Delta u=0 & \text{in } \Omega \text{ and } t>0,\\ u=0 & \text{on } \Gamma \text{ and } t>0,\\ u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x) & \text{in } \Omega,\end{cases}$$

where Ω is an exterior domain in an $n-$ dimensional Euclidean space $\mathbb{R}^n$, whose boundary Γ is a C^∞ and compact hypersurface. Below, $r_0>0$ is a fixed constant such that $\Omega^c\subset B_{r_0}=\{x\in\mathbb{R}^n \mid |x|<R\}$. ($\Omega^c$ is the complement of Ω.)

In the wave equation case, the local energy decays exponentially fast if n is odd and polynomially fast if n is even, when Ω is at least non–trapping (cf. [9], [10], [11], [16]). In fact, from a physical point of view the energy propagates along the wave fronts, so that the motion stops after time passes unless the wave front is trapped in a bounded set.

In the dissipative wave case, the energy also propagates along the wave front. Moreover, the trapped energy also decreases in virtue of the dissipative term u_t, so that we can expect to get the local energy decay result for any domains. In fact, in 1983 Shibata [14] proved the following theorem.

Theorem 1.1. *Assume that $n\geq 3$. Let $R>r_0$ and let $u(t,x)$ be a smooth solution of (D) such that $suppu(0,x)$, $suppu_t(0,x)\subset\Omega_R=\{x\in\Omega \mid |x|<R\}$. Then, there exists*

a constant $C > 0$ depending on n and R such that

$$\int_{\Omega_R} \{|u_t(t,x)|^2 + \sum_{|\alpha|\le 1} |\partial_x u(t,x)|^2\} dx$$
$$\le C(1+t)^{-n}\{\sum_{|\alpha|\le 3}\int_\Omega |\partial_x^\alpha u_t(0,x)|^2 dx + \sum_{|\alpha|\le 4}\int_\Omega |\partial_x^\alpha u(0,x)|^2 dx\},$$

where $\partial_x^\alpha v = \partial^{|\alpha|} v/\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1,\dots,\alpha_n)$ *and* $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The purpose of this study is to show the decay rate of the local energy of even the weak solutions of (D) is also $n/2$ when $n \ge 2$, that is, we shall prove the following theorem.

Theorem 1.2. *Assume that $n \ge 2$. Let $R > r_0$ and $u_0 \in H^1_{0,R}(\Omega)$ and $u_1 \in L^2_R(\Omega)$, where*

$$L^2_R(\Omega) = \{f \in L^2(\Omega) \mid supp f \subset \Omega_R\},$$
$$H^1_{0,R}(\Omega) = \{f \in H^1(\Omega) \mid supp f \subset \Omega_R,\ \ f = 0 \text{ on } \Gamma\}.$$

Let $u(t,x)$ be a weak solution of (D). Then, there exists a constant C depending on n and R such that

$$\int_{\Omega_R} \{|u_t(t,x)|^2 + \sum_{|\alpha|\le 1} |\partial_x^\alpha u(t,x)|^2\} dx$$
$$\le C(1+t)^{-n}\{\int_\Omega |u_1(x)|^2 dx + \sum_{|\alpha|\le 1}\int_\Omega |\partial_x^\alpha u_0(x)|^2 dx\}.$$

Compared with Theorem 1.1, in Theorem 1.2 we remove the smoothness assumption on solutions of (D) and we consider the case that $n = 2$ as well as the case that $n \ge 3$.

For the Cauchy problem of the dissipative wave equation (i.e. $\Omega = \mathbb{R}^n$), A. Matsumura [8] studied the decay rate of solutions in 1976. His argument was based on the concrete representation of solutions by using the Fourier transform. When Ω is bounded, it is well–known that the energy of solutions decays exponentially fast. In fact, this fact is easily proved by the multiplications of the equation with u_t and u

and by use of Poincaré's inequality. Since Ω is unbounded in our case, we cannot use Poincaré's inequality. And also, because of the boundary, we can not use the Fourier transform. Our method is based on a spectral analysis to the corresponding stationary problem.

§2. A construction of C_0 semigroup solving (D).

Putting $u_t = v$, let us rewrite the problem (D) in the following form :

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}.$$

To consider A to be dissipative, we introduce a space $H_D(\Omega)$. For any open set $\mathcal{O} \subset \mathbb{R}^n$, $C_0^\infty(\Omega)$ denotes the space of all C^∞ functions on $\mathbb{R}^n$ whose support is compact and lies in $\mathcal{O}$ (in particular, such functions vanish near the boundary of $\mathcal{O}$), $L^2(\mathcal{O})$ a usual L^2 space on $\mathcal{O}$ with norm $\|\cdot\|_{\mathcal{O}}$ innerproduct $(\ ,\ )_{\mathcal{O}}$ and $H^s(\mathcal{O})$ a usual Sobolev space of order s on $\mathcal{O}$ with norm $\|\cdot\|_{s,\mathcal{O}}$. $\|\cdot\|_{k,\Omega}$ will be denoted simply by $\|\cdot\|_k$. Likewise for $\|\cdot\|_\Omega$ and $(\ ,\ )_\Omega$. Then, we put

$$\begin{aligned} H_D(\Omega) =&\{u \in H^1_{loc}(\Omega) \mid \nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) \in L^2(\Omega),\ \ u = 0 \ \ \text{on } \Gamma, \\ &\exists\ \{u_n\} \subset C_0^\infty(\Omega) \text{ s.t. } \|\nabla(u_n - u)\| \to 0 \text{ as } n \to \infty\ \}, \end{aligned}$$

where $H^1_{loc}(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid u \in H^1(\Omega_R)\ \forall R > r_0\}$. $H_D(\Omega)$ has the following properties.

Theorem 2.1. *If $u \in H_D(\Omega)$, then u satisfies the following inequalities:*

$$\|u\|_{0,\Omega_R} \le C(R)\|\nabla u\|_{0,\Omega_R},$$
$$\int_\Omega \frac{|u(x)|^2}{d(x)^2}dx \le C\|\nabla u\|^2.$$

Moreover, $H_D(\Omega)$ is a Hilbert space equipped with an inner product $(u,v)_D = (\nabla u, \nabla v)$.

Then, an underlying space for A is

$$\mathcal{H} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \mid u \in H_D(\Omega) \;\; v \in L^2(\Omega) \right\}.$$

From Theorem 2.1 we know that $\mathcal{H}$ is a Hilbert space equipped with the innerproduct

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \right)_{\mathcal{H}} = (u, w)_D + (v, z).$$

The domain of A is

$$\begin{aligned} D(A) =& \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid A \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \right\} \\ =& \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H} \mid v \in H_D(\Omega), \;\; \Delta u \in L^2(\Omega) \right\}. \end{aligned}$$

Then, A has the following properties.

Proposition 2.2. *(1) A is a closed operator. (2) A is a dissipative operator. (3) $\mathcal{R}(I - A) = \mathcal{H}$. (4) $D(A)$ is dense in $\mathcal{H}$.*

Lumer and Phillips theorem [13, Chapter 1, Theorem 4.3] implies that A generates a C^0 semigroup $\{T(t)\}$ on $\mathcal{H}$.

§3. A proof of Theorem 1.2.

Our purpose in this section is to prove the following result, which implies our main theorem.

Theorem 3.1.

$$\|\varphi_R T(t)\mathbf{x}\|_{\mathcal{H}} \le C(1 + t)^{-n/2} \|\mathbf{x}\|_{\mathcal{H}},$$

for $\mathbf{x} \in \mathcal{H}_{1,R}$, where $C = C(R)$.

Sketch of proof.

Since A is dissipative, $T(t)$ is a C_0 semigroup of contractions, so that

$$\|T(t)\| \leq 1 \quad \forall t \geq 0. \tag{3.1}$$

Let α be a positive number. In view of (3.1), we have the following expression :

$$T(t)\mathbf{x} = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\alpha-i\omega}^{\alpha+i\omega} e^{\lambda t}(\lambda I - A)^{-1}\mathbf{x}d\lambda \quad \text{for } \mathbf{x} \in D(A^2). \tag{3.2}$$

(cf. [12, p.295] or [13, Chapter 1, Corollary 7.5]). By a lemma due to F. Huang in [4, §1, Lemma 1] (also see [7]), we have the following lemma.

Lemma 3.2. *For any $\alpha > 0$ and $\mathbf{x} \in \mathcal{H}$, put*

$$g(\omega) = \|((\alpha + i\omega)I - A)^{-1}\mathbf{x}\|_{\mathcal{H}}.$$

Then $g(\omega) \in L^2(\mathbb{R})$ and

$$\lim_{|\omega|\to\infty} g(\omega) = 0,$$
$$\int_{-\infty}^{\infty} g(\omega)^2 d\omega \leq \frac{\pi}{\alpha}\|\mathbf{x}\|_{\mathcal{H}}^2.$$

In view of Lemma 3.2, the high frequency part decays sufficiently fast, so that we have to investigate the low frequency part. Now we shall introduce some functional spaces. Let E be a Banach space with norm $|\cdot|_E$, $N \geq 0$ an integer and $k = N + \sigma$ with $0 < \sigma \leq 1$. Put

$$\mathcal{C}^k(\mathbb{R}^1; E) = \{u \in C^{N-1}(\mathbb{R}^1; E) \cap C^\infty(\mathbb{R}^1 - \{0\}; E);\ \ll u \gg_{k,E} < \infty\},$$

where

$$\begin{aligned} \ll u \gg_{k,E} = &\sum_{j=0}^{N} \int_{\mathbb{R}} |(\frac{d}{d\tau})^j u(\tau)|_E d\tau \\ &+ \sup_{h\neq 0}|h|^{-\sigma} \int_{\mathbb{R}} |(\frac{d}{d\tau})^N u(\tau + h) - (\frac{d}{d\tau})^N u(\tau)|_E d\tau \quad \text{if } 0 < \sigma < 1, \end{aligned}$$

$$\ll u \gg_{k,E} = \sum_{j=0}^{N} \int_{\mathbb{R}} |(\frac{d}{d\tau})^j u(\tau)|_E d\tau$$

$$+\sup_{h\neq 0}|h|^{-1}\int_{\mathbb{R}}|(\frac{d}{d\tau})^N u(\tau+2h)-2(\frac{d}{d\tau})^N u(\tau+h)+(\frac{d}{d\tau})^N u(\tau)|_E d\tau,$$

if $\sigma=1$. Here, $\left(\frac{d}{d\tau}\right)^0=1$. The following lemma is concerned with the properties of the Fourier transformation of functions belonging to $\mathcal{C}^k(\mathbb{R}^1,E)$, which was proved in [14, Part 1, Theorem 3.7].

Lemma 3.3. *Let E be a Banach space with norm $|\cdot|_E$. Let $N\geq 0$ be an integer and σ a positive number ≤ 1. Assume that $f\in\mathcal{C}^{N+\sigma}(\mathbb{R}^1;E)$. Put*

$$F(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\tau)\exp(\sqrt{-1}\tau t)d\tau.$$

Then,

$$|F(t)|_E\leq C(1+|t|)^{-(N+\sigma)}\ll f\gg_{N+\sigma,E}\cdot$$

Here and hereafter, we put $\mathcal{H}_R=\{\begin{bmatrix}u\\v\end{bmatrix}\in\mathcal{H}\mid \text{supp } u,\ \text{supp } v\subset\Omega_R\}$. $\varphi_R(x)$ always refers to a function in $C_0^\infty(\mathbb{R}^n)$ such that $\varphi_R(x)=1$ if $|x|\leq R$ and$=0$ if $|x|\geq R+1$. Moreover, we put

$$\mathcal{H}_{loc}=\left\{\begin{bmatrix}u\\v\end{bmatrix}\mid u\in H^1(\Omega_R),v\in L^2(\Omega_R)\ \forall R\geq r_0\right\},$$

$$\mathcal{H}_{comp}=\bigcup_{R\geq r_0}\mathcal{H}_R,$$

and $\mathcal{L}(B_1,B_2)$ denotes the set of all bounded linear operators from B_1 into B_2 and $Anal(I,B)$ the set of all B–valued analytic functions in I. In view of Lemma 3.3, if we prove the following fact, the proof of Theorem 3.1 is complete.

(F) Put $Q_d=\{\lambda\in\mathbb{C}\mid 0<\Re\lambda<d,\ |\Im\lambda|<d\}$. Then, there exists a $d>0$ and $R(\lambda)\in Anal(Q_d;\mathcal{L}(\mathcal{H}_{comp},\mathcal{H}_{loc}))$ such that :

(a) $\quad R(\lambda)\mathbf{x}=(\lambda I-A)^{-1}\mathbf{x}\quad$ for $\mathbf{x}\in\mathcal{H}_{comp}$ and $\lambda\in Q_d$;

(b) For any $R\geq r_0$ and $\rho(s)\in C_0^\infty(\mathbb{R})$ such that $\rho(s)=1$ if $|s|<d/2$ and $=0$ if $|s|>d$, there exist $M_1>0$ depending on R, ρ and d such that

$$\ll\rho(\cdot)(\varphi_R R(\alpha+i\cdot)\mathbf{x},\mathbf{y})_{\mathcal{H}}\gg_{n/2,\mathbb{R}}\leq M_1\|\mathbf{x}\|_{\mathcal{H}}\|\mathbf{y}\|_{\mathcal{H}},$$

for any $\mathbf{x} \in \mathcal{H}_R$, $\mathbf{y} \in \mathcal{H}$ and $0 < \alpha < d$.

We shall conclude this report by giving a brief proof of (F).

Proof of (F).

When $n \geq 3$, (F) was proved by Shibata [14, Part 1], so that we shall consider the case that $n = 2$. Corresponding stationary problem is

$$(\lambda^2 + \lambda - \Delta)u = f \quad \text{and} \quad u = 0 \quad \text{on } \Gamma. \tag{3.3}$$

If $|\lambda|$ is small, then in stead of (3.3), it is sufficient to consider the following problem :

$$(\lambda - \Delta)u = f \quad \text{in } \Omega \subset \mathbb{R}^2 \quad \text{and} \quad u = 0 \quad \text{on } \Gamma, \tag{A_λ}$$

where $\lambda \in S_{r,\varepsilon} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\lambda| < r,\ |\arg\lambda| < \pi - \varepsilon\}$, $0 < r < 1$ and $0 < \varepsilon < \pi/2$, because $\lambda^2 + \lambda$ is equivalent to λ for small $|\lambda|$. In view of Lemma 3.4 of [14], in order to prove (F), it is sufficient to prove the following propositions.

Proposition 3.4. *For $\lambda \in S_{r,\varepsilon}$ and $r_0 \leq R < \infty$, there exists $A(\lambda) : L^2_R(\Omega) \to H^1_{loc}(\Omega)$ satisfying that*

$$(\lambda - \Delta)A(\lambda)f = f \quad \text{in } \Omega \quad \text{and} \quad A(\lambda)f = 0 \quad \text{on } \Gamma,$$

for $f \in L^2_R(\Omega)$. Moreover, it satisfies that

$$\|\varphi_R A(\lambda)f\|_1 \leq C\|f\| \quad \text{as } \lambda \in S_{r,\varepsilon}.$$

Proposition 3.5. *For λ and R as mentioned above, following estimates hold;*

$$\|\varphi_R \frac{d}{d\lambda} A(\lambda)f\|_1 \leq \frac{C(R)}{|\lambda||\log\lambda|^2}\|f\| \leq \frac{C(R)}{|\Im\lambda|}\|f\|,$$
$$\|\varphi_R \frac{d^2}{d\lambda^2} A(\lambda)f\|_1 \leq \frac{C(R)}{|\lambda|^2|\log\lambda|^2}\|f\| \leq \frac{C(R)}{|\Im\lambda|^2}\|f\|,$$

for $f \in L^2_R(\Omega)$.

Our main idea to prove Propositions 3.4 and 3.5 is to use the single layer potential and the double layer potential, and to reduce (A_λ) to a boundary integral equation. Put $v = (\lambda - \Delta)^{-1} f$. Then v is represented by the modified Bessel function :

$$(\lambda - \Delta)^{-1} f = \int_{\mathbb{R}^2} E_\lambda(x - y) f(y) dy,$$

where $E_\lambda(x) = (2\pi)^{-1} K_0(|x|\sqrt{\lambda})$, K_m ($m \in \mathbb{N} \cup \{0\}$) denotes the modified Bessel function of order m. So we want to solve the equation

$$(A'_\lambda) \qquad (\lambda - \Delta) w = 0 \quad \text{in } \Omega \quad \text{and} \quad w = f_\lambda \quad \text{on } \Gamma,$$

where $f_\lambda = (\lambda - \Delta)^{-1} f|_\Gamma$. To do this, let us introduce the integral operator B_λ :

$$B_\lambda \Phi = D_\lambda \Phi - \eta E_\lambda M \Phi + \frac{2\pi\alpha}{\log\sqrt{\lambda}} E_\lambda \Phi \quad \text{for } \Phi \in C^0(\Gamma).$$

Here α, $\eta > 0$, E_λ is a single layer potential defined by

$$E_\lambda \Psi(x) = \int_\Gamma E_\lambda(x - y) \Psi(y) do_y$$

and D_λ is a double layer potential defined by

$$D_\lambda \Psi(x) = \int_\Gamma D_\lambda(x, y) \Psi(y) do_y,$$

where

$$\begin{aligned} D_\lambda(x, y) &= \nabla_x E_\lambda(x - y) \cdot N(y) \\ &= -\frac{1}{2\pi} K_1(|x - y|\sqrt{\lambda}) \frac{\sqrt{\lambda}}{|x - y|} (x - y) \cdot N(y). \end{aligned}$$

The projection $M : C^0(\Gamma) \to C^0(\Gamma)$ is defined by

$$\Phi \to M\Phi = \Phi - \Phi_M \quad \text{with } \Phi_M = \frac{1}{|\Gamma|} \int_\Gamma \Phi do \text{ and } |\Gamma| = \text{meas}(\Gamma).$$

Obviously $B_\lambda\Phi$ satisfies that $(\lambda-\Delta)B_\lambda\Phi=0$ in Ω, so that we obtain the following boundary integral equation :

$$B_\lambda\Phi|_\Gamma = K_\lambda\Phi = \big(-\frac{1}{2}+D_\lambda-\eta E_\lambda M+\frac{2\pi\alpha}{\log\sqrt{\lambda}}E_\lambda\big)\Phi = f_\lambda. \tag{3.4}$$

If Φ is a solution of (3.4), $B_\lambda\Phi$ satisfies (A'_λ), and $A(\lambda)f$ is expressed by

$$A(\lambda)f=(\lambda-\Delta)^{-1}f-B_\lambda\Phi. \tag{3.5}$$

Therefore, (A_λ) was reduced to a boundary integral equation (3.4). K_λ is a Fredholm operator, so that by using the Fredholm alternative theorem, we can solve the boundary equation (3.4). If we consider that $A(\lambda)$ is an operator from $L^2_R(\Omega)$ to $L^2_{loc}(\Omega)$, by the properties of Bessel function, we know that the expansion of $A(\lambda)$ at $\lambda\to 0$ is

$$A(\lambda)=C_0+C_1\frac{1}{\log\lambda}+\cdots.$$

Therefore, we have Propositions 3.4 and 3.5.

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